**Path Integral Methods for GF, Z**

So let’s look at formulating a path integral representation of the GF, specifically GC\*. Getting GC\* in terms of path integral is easier in respects than getting the other non-equilibrium GF path integral representations. This is because the we don’t have to separate out the true GS from our trace. Rather we’re summing over all all states. And we’ll also look at path integral formulations of Z, and Ξ. Other path integral methods for Z will be explored in the Quantum MFT files.

**Single particle**

Note that we start out just calculating Z, but including correlations, i.e. G, should be in your mind as well. Suppose that we start with a Hamiltonian



And we’d like to obtain the partition function for this guy.



We can evaluate the trace by using position eigenstates – though other bases are possible as well.



Concentrating on the integrand, chopping e-BH up into intervals (and see the QM Time-Dependent/Propagator as Path Integral file for why can say e-β(H\_0+V) = e-βH\_0e-βV), and basically copying the work from the path integral stuff in the QM folder, we’ll have (implicit summation over all X’s, P’s):



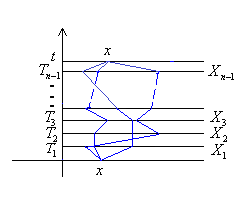
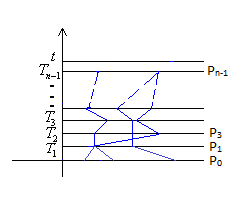
Using τ = it, we can convert the momentum wavefunction things to real exponents,



And this would come to, similarly to before,



where x-dot = dx/dτ, and the paths we’re integrating over begin and end at the same point,

while the momentum path integral has no restrictions. What we get, after integrating out the p-stuff is (see Path Integrals file, or recall ∫dx exp(-(ax2 + bx + c)) = √(π/2a)exp(b2/4a)) :



which is just,



which is just an integration over the (complex time) action from 0 to β and over all positions, starting at x(0) = x, and ending at x(β) = x as well. Continuing, we can write the action as:



Anyway, so we have:



This way of writing it doesn’t seem to prevalent. Maybe another way to write this is as:



which will mean to sum over all closed paths, regardless of where they start. These two expressions are equivalent. But while we know how to compute the former using all that GP methodology, how do we compute the latter, w/o recourse to the former? I guess one way is to make a Fourier decomposition of the paths, x(τ), over functions periodic over interval β. And then do the integration over those independent d.o.f. It would also seem that we can use the usual formulas for path integrals, with the caveat that the GF (A-1), obey periodic boundary conditions. Clearly I need more mathematical rigour here, and elsewhere, when dealing with these path integral things, to properly distringuish what they mean and how to calculate them. But don’t got time for that.

Let’s do two quick examples, using the former expression for Z. We can look at the free particle partition function, in 1D, borrowing our result from QM folder:



as compared to, from the Stat. Mech. file



where in the last we set ℏ = 1, which introduces the extra factor of 2π. So we see they match. Let’s try the HO.



which matches what I got in the stat mech file. Now we’ll consider the (a) Green’s function.



Given the stuff above, and the work in the QM file, and QFT folder, etc., it seems plausible at this point that we can write the GF as:



Higher order GF’s would be written similarly.

**Many (distinct) particle**

Presuming distinguishable particles at present we can write, analogously to above,



or in other words,



where **x** is the (3)N-dimensional vector describing the positions of all particles. And,



Now we’ll consider the (a) many-body Green’s function on the lattice. This is:



Given the stuff above, and the work in the many-body file, it seems plausible at this point that we can write the many-particle GF as:



where **x** is now just a 3 dimensional vector. Higher order GF’s would be written similarly.

**Many (identical) particle**

Might want to see the QM/Many Identical Particles/Propagator Path Integral file, and QM/Time-Dependent/HO Coherent States file for relevant background. Now let’s consider the many-body Hamiltonian for identical particles, specifically bosons, so i don’t have to deal with spin, but the generalization should be clear enough:



and we’ll have:



where |ψσ,a(x)> is some initial eigenstate (a means initial) of the fermion field operator(s) σ(x), with Grassman eigenvalues ψσ,a(x). And where DN[ψσ,a\*(x)ψσ,a(x)] = Πx∈V dψ↑a\*(x) dψ↑a(x)dψ↓a\*(x) dψ↓a(x) signifies integrating over all types of spins’ field values at all points in space, but restricting ourselves to fields with a constant particle number N. I’ve obviously presumed we would take the trace of e-βH with creation/annihilation operator eigenstates. We could use position operator eigenstates instead. But this is the usual route. Note the analogy with the single particle case:



Okay, now let’s break that expectation down. So consider:



Recalling the identity we derived, more or less, in the QM many-identical particle propagator file (and before that, in the QM Time-Dependent HO coherent state file)



and updating to encompass our up/down spin types, we’ll have:



And now we’ll group the overlap terms together, and the H’s together, and make changes like ψσ,a\* → ψα,n-1\* in the top line, ψσ,n-1\* → ψα,n-2\* in the second line, etc., valid to δτ, as was done in the propagator file, though its been pointed out that Grassman numbers seem to lack a ‘measure’ of magnitude, which puts this operation’s validity in doubt. But whatever…



and now we can introduce derivatives,



Taking the continuum limit, we can write this as a functional integral, and we have:



where the integration bounds are to indicate that we have the boundary conditions ψσ(x,0) = ψσ,a(x) and ψσ(x,β) = ψσ,a(x). And so we can write:



or in other words,



where,



An integration by parts on say the first term, will make this:



The result is indistinguishable between bosons and fermions, formally. But the distinction comes in the boundary conditions obeyed by the GF, which follows (G = A-1):



The boundary conditions will enforce that A-1 be symmetric in time over interval (0,β) if bosons, and anti-symmetric in time over interval (0,β) if fermions. Not sure how to reconcile that with the fact that the trace seems to necessitate periodic boundary conditions regardless, but I’ll chalk it up to the weirdness of Grassman numbers.

The restriction to constant particle number fields would seem impractical. We can lift the restriction, but then we would have to calculate the grand canonical partition function instead. So we can say,



where,



and K = H – μN of course. So basically, we just subtract μ from the kinetic energy term in the action. Now we’ll consider the (a) many-body Green’s function.



Given the stuff above, and the work in the many-body file, it seems plausible at this point that we can write the many-(identical) particle GF as:



For fermion fields, actually, we’d usually write instead of ψ\*. Higher order GF’s would be written similarly.

***example***

Let’s calculate the free fermion partition function. This is,



where,



Formally, our answer is (see Path Integrals folder):



To explicitly evaluate this, it’s best to go to the eigenbasis of the operator. So we want to examine,



with implicit boundary conditions of periodicity over the volume of our sample, and also anti-periodicity in time between 0 and β (fermions are anti-periodic while bosons are periodic, as we’ll recall). Well we know that the solutions are:



So we have (might see Path Integrals file for more explicit approach):



Therefore,



Need to do that frequency sum. So recall from the Math Appendix that identity,



But doesn’t look like the integral f(z)nF(z) will converge as |z| → ∞. Maybe if we take a derivative w/r to μ, say. Then we’d have:



Then this should converge, and we won’t need a branch cut, since we don’t have that ln anymore. So,



So,



which we know is the particle number, so it’s working out. Then integrating,



Obviously our result is just the first term. Not sure how we’d ascertain that though. Well, let’s say we go back to the original formula.



and use the identity,



and ignore the convergence issues at |z| → ∞. Also, the ln(z) function has no poles, and we’ll have to integrate around the branch cut Re(z) > k2/2m­ – μ. Contour looks like this, but infinitely large.

Chart, diagram

Description automatically generated with medium confidence

Then we have, using the principle value of the ln(z) function [ln(x+i0+) = ln|x|, and ln(x – i0+) = ln|x| + 2πi0+]:



Well, doing the same manipulations as before,



which we can write as:



So we finally have the (unequivocally) correct answer. I guess we can just ignore the convergence issues at the boundary of the contour.

**Bosonic field partition function**

Let’s consider the case of relativistic bosons, with an extra potential V(φ) describing anharmonic terms perhaps. In that case our H is:



Then the partition function would be:



or rather,



where,



and Hφ is just the part of H without the π field. Accordingly, the action simplifies to:



Now we’ll consider the (a) many-body Green’s function.



Given the stuff above, and the work in the QFT file, it seems plausible at this point that we can write the (a) field GF as:



Higher order GF’s would be written similarly.