**Navier-Stokes equation**

Now I’d like to derive the Navier-Stokes equations. In our fluid, we won’t have impurities obstructing flow. So our local equilibrium distribution function should have well defined values of n, **℘** (non-zero this time, thanks to no impurities), ε. And I’ll suppose that we allow having a fluid flowing in a conservative force field ψ (which is part of the energy ε), perhaps also subject to some non-conservative force Fnc so that F = -q∇ψ + Fnc. So we’ll consider s(ε,**℘**,n,ψ) (**℘** shows up now, unlike in the previous file, because we don’t presume to have impurities which would try to ‘relax’ the velocity distribution of any particular slice back to zero). I’ll ignore angular momentum contributions as a liquid cannot exert shear forces I don’t believe. And I’ll ignore internal structural energy changes. So then I’ll say the balances are:



By the way, ∇·(**℘v**) = ∇j·(℘ivj) (using Einstein summation notation) and ∇·**π** = ∇jπij = ∇jπji (‘cause π is a symmetric tensor), and ∇·(**π**·**v**) = ∇iπijvj = ∇jπijvi, again, ‘cause **π** is a symmetric tensor. Plugging the balances into the entropy balance, using (see Potentials file) Ts = (ε + p - ℘v – μn), and ds = (1/T)dε – (v/T)·d℘ - (μ/T)dn – (m/T)dψ (where m = qn in our case, q being say, mass or charge, depending on the field we have present). Using Einstein summation notation by the way:



Now solving for sint…and simplifying,



where I changed indices in one term i 🡪 j and j 🡪 i. Then continuing, can see those ℘ terms cancel as well. This leaves us with:



Or back in vector language,



In a previous file, Thermo/NETD continuum transport, we neglected the term (π-p)∇v. But that was because we had a situation (had impurities basically) which made v a 1st order perturbed quantity, and consequently ∇v a 2nd order quantity, and hence (π-p)∇v a 3rd order quantity and rightfully neglected. But here, we expect v to be part of the equilibrium state, and so a 0th order quantity, and so then (π-p)∇v is 2nd order and therefore appropriate to keep.

**Onsager’s relations**

In order for sint to be positive definite, we must postulate some connection between the currents and the gradients. A linear relationship is typically assumed. And the constants of proportionality are Onsager’s coefficients. So we’d say, in general,



So we can say that generically,



Onsager hypothesized that the matrix is symmetric, which has been verified so far, at least close to equilibrium which is where the theory works. Secondly some coefficients can be known to be zero, by Curie’s principle that higher symmetry forces cannot drive lower symmetry currents.

**Dissipationless fluid flow**

Now we consider the case where dissipation is zero. I believe this means that we don’t have any irreversible processes.



and these equations do indeed result in annihilation of the σ term. Filling this into the prior conservation equations, we would have:



These two equations can be combined to give:



and this is Euler’s equation:



If we integrate this equation along a streamline, we would get Bernoulli’s equation – see Classical Mechanics folder (or next file). We could plug the energy current into the energy equation and see what we get there.



and then we have three equations and three unknowns: n, v, T, since p = p(n,T) and ε = (1/2)mnv2 + u where the internal energy u = u(n,T).

**Dissipative Fluid Flow**

On the other hand, if we allow dissipation then following Onsager’s hypothesis, the currents ought to be proportional to the thermodynamic forces, so that (using Einstein summation notation)



Now for an isotropic fluid, the L’s ought to be isotropic. This requires L1 = κ. And L3 must be isotropic. An isotropic tensor is one which looks the same under any rotation – i.e., they should be invariant with respect to any rotation. Now any scalar is an isotropic function. In order for a vector to be isotropic, we’d need the following. Consider a vector **v**.



(Einstein summation notation) Upon rotation of the vector we will get:



where  I s the vector that  get’s rotated onto. If **v** is to be isotropic, then this must give us the same vector back. So we need:



Suppose we do a 90 degree rotation about the z-axis. Then this equation would read:



which requires



So our vector is now:



if we do a 90 degree rotation about the x-axis, then the isotropy condition would be:



which requires that v3 = 0. So there is no isotropic vector. Basically this is because if you rotate a vector by any amount, it doesn’t equal itself again (except for special values of rotation to be sure but for a vector to be isotropic it must rotate onto itself for all possible rotations). Let’s now consider rank two tensors.



Upon rotation we’d have:



And if it’s isotropic then we want,



Suppose we do a 90 degree rotation about the z-axis. Then this would read:



which requires,



If we do a rotation about the x-axis, and y-axis similarly we will generate equations that require all off-diagonal elements to be zero, and all diagonal elements to be identical. So the most general rank 2 tensor that is isotropic will be found to be:



Now let’s go to rank three. The most general rank three tensor is:



And upon rotation we get:



We can perform similar arguments as before. Rotate the axes in various directions and determine what the  are. Then equate **M** and **M**′. We will come to the conclusion that all elements of **M** must be zero. Going to rank four, the most general tensor would be:



and upon rotation would be:



Performing the same arguments, we would arduously determine that the most general tensor that can be constructed is just products of rank 2 isotropic tensors, namely that:



Finally, L3 should be symmetric w/r to permutations (ij) and (kℓ) – I believe this is related to conservation of angular momentum). Which means



These equations demand that b = c. So we have:



Finally we will write this as:



λ is called the bulk viscosity, and η is the shear viscosity. So this gives us an important equation for the surface force, **π**. We have:



and so we have:



Now let’s plug these currents into the conservation equations:



and,



Substituting the first into the second gives:



Simplifying…



and altogether we have:



If we add to it the energy equation, which I’m not bothering to do, then we’ll have, as before, three equations and three unknowns. Of course we don’t a priori know the transport coefficients η or λ. But these can be measured, or, we can use NESM to estimate them (and we do – see that Stat Mech/viscosity file).