**Relaxation Time Approximation**

**Diffusion from RTA Mean Field Solution**

Let’s examine the diffusivity from the perspective the MFT solution to the RTA equation, which we developed in a previous file. So we found these two self-consistent equations:



where D was given by:



Let’s consider the time-independent case, and presume no external fields. So this is pure diffusion. Then we have:



Does crossing out the τsc∂j/∂t term mean that τsc is small, and therefore we have fast thermal equilibration? I guess. Now divide both sides by e to get the number current density,



From the continuity, equation, we have:



We can combine these equations to get an equation for the density alone. So take the divergence of the first equation, presuming **D** is an isotropic tensor (**D** = D**1**)



and plug into the continuity equation, to get the diffusion equation.



Can solve this equation. Let’s presume spherical symmetry,



And we’ll take our initial condition to be the typical illustrative one: N particles all packed into the point r = 0, i.e., n0(r) = Nδ(r). Taking Laplace transform on time, we have:



For r ≠ 0, we have:



Let = r. Then,



So,



We don’t expect the density to ever exponentially grow with radius (at fixed time). So we can cross out the B term.



Then we have to determine A. We can determine this by employing the delta function thing. So let’s put our solution back into the full equation,



where in the last line we use Gauss’s law. Filling in our result,



Taking the ε → 0 limit,



So then our solution for is:



Now have to take inverse Laplace transform to get n(r,t). So,



Will deform the contour to wrap around the negative real s axis,



Okay so now we have a real integral to do,



Well let’s change variables s → s2.



and note that the following integral is the same, changing variables s → -s,



So we can add the two versions of the same integal together, and divide by two, to get:



Now we’ll integrate by parts,



And so we have:



This describes a Gaussian which is infinitely peaked at t = 0 (that’s the delta function initial condition), and spreads out over time. The standard deviation of Gaussian is:



and this gives us the rough radius encompassed by our particles as a function of time. Might check out the Brownian motion files later in this folder for comparison. Now let’s go back to our equation,



and take out the fields, and the explicit time-dependence



We’ll note that there is a harmonic solution to this equation. Let j(r) = Re[j(q)eiq·r], and n(r) = Re[n(q)eiq·r]. Filling these in, we have:



If we presume the diffusion constant is isotropic tensor **D** = D**1**. Then we have:



where ρ = en. We can generalize to time-dependent density fluctuations too, and say j(r,t) = Re[j(q,ω)eiq·r-iωt], n(r,t) = Re[n(q,ω)eiq·r-iωt], and j(q,ω) = -eD(q,ω)n(q,ω). But this time, we’ll need to use the time-dependent RTA equation (still no fields),



So plugging everything in,



Again, let’s presume an isotropric diffusion tensor **D** = D**1**. Then we have:



where ρ = en. And so we have an effective diffusion constant,



**Angular Diffusivity**

We can talk about diffusion in the context of electric dipoles too. We looked at this problem in the Thermodynamics folder/Brownian motion. We’ll start with the RTA equation for a system of particles defined by their orientation, **θ**, and angular momentum **S**. Making the correspondance r → θ, m → I, k → S, we can write down the evolution equation for the distribution function f(**θ**,**S**), which gives the probability a particles is in the state (**θ**,**S**).



where I is the moment of inertia (we’ll just presume spheres for simplicity) of the particles, and T is the torque on our particles. Now we’ll reproduce the steps we took above. First we’ll look at the continuity equation. So first let’s define the number density,



Note **S**/I = **ω**, and so **j** is describing how many particles are flowing into the angle **θ**. This is an abstract kind of ‘flowing’, as it’s not flowing through physical space, but through ‘angle’ space. We can/will also define an average (entropic) angular velocity as:



And if we multiply by N and integrate both sides of the Boltzman equation over d3S we’ll get:



The RHS gives us RHS = [nleq(θ) – n(θ)]/τsc. In fact, we should get zero. This is because really, we should be using a more sophisticated approximation for the RHS: the collision integral. We’ll do this for f(r,k), but not for f(θ,S). So we’ll just have to accept this by analogy. So we’re left with:



And in terms of the average angular velocity defined above, we have:



But I guess I’ll leave it as:



Now let’s multiply both sides by N**S**, and integrate over d3S,



This time our RTA approximation will be acceptable for the RHS (can see NESM balance equations again). The integral over fleq will be zero, as fleq is symmetric in S (kinetic energy is S2/2I). But the integral over f will not be zero, generally. And I’m going to make our torque explicit.



where p is the dipole moment of the dipoles, and E is the electric field. Then we can say,



Now ∫d3S N**S** f(θ,S) is just the **S**×(number) current density, **j**(θ) = n(θ)**ω**(θ) where n(θ) is the local (in angle-space) particle density, and **ω**(θ) is the local (in angle-space) average velocity of the particles. Can/will also write this as **j**(n)(θ). Now we have:



Gonna do the two integrals component by component, using Einstein summation notation. First guy is:



I think at this point we’ll introduce a diffusion tensor. Let’s say,



If we approximate D by evaluating it via a local equilibrium distribution (assuming we can use classical statistics), we can say,



So,



But regardless, then we have:



I guess I’ll take D to be isotropic, as I kind of explicitly did in the highlighted formula,



and then in vectors we’ll have:



And the other guy is:



where we IBP, and take the boundary term to go to zero, as it should for large S, since f(S) → 0. So going back to vectors, we can say,



So altogether, we have:



So our two equations are:



Now in parallel with our manipulations above, we’ll take j to be slowly varying with time, so that we can neglect ∂j/∂t. And we’ll take the divergence of the bottom equation and plug it into the first. This will give us:



Now E is presumably a constant, but p would depend on the orientation, θ, of the dipole. So I’ll write **p**(θ). Then we have:



This is called Debye’s equation.