**Noninteracting Green’s Functions**

Now we’ll examine some non-interacting GF’s. Most are of the same form as those obtained before/only difference being a thermal averaged expectation replacing a single state expectation. Note I’m going to be presuming time-development starts at t = 0: So U0(t) = e-iH0·t. But if we used U0(t,t0) = e-iH0(t-t0) instead, this wouldn’t change G(t,t´) because it is, as you’ll see, a function only of the difference of the arguments. Also, for the most part, I won’t bother with the exponential convergence factors that we should have on the real space GF’s. Might want to check out QM/Many Particles/Identical Particles for more work on these, especially apropos the Fourier transforms.

**Free (Identical) Particle Green’s Functions**

Consider now the case of independent particles with eigenstates, energy levels: φj, εj. Then



And with the time development operator U = e-iKt, we have:



where ξλ = ελ – μ, recall. Then the complex time GF is, in energy space,



where nj is the fermionic/bosonic occupation number formula. The FT is:



and so we have,



Now let’s go backwards. Taking the inverse transform,



To perform the summation we’d use the Residue theorem from Complex variables (see Math Appendix)



So we have:



Note we cannot apply the residue theorem for all τ because it requires that the integral of,



be zero around the contour. Well it looks something like this for negative/positive τ.



So when τ < 0, it always goes to zero, because one of the two terms in the denominator will be very large. But it blows up for large negative z when τ is positive. Well, let’s suppose that τ is negative for now. Then we can do the usual contour integration because the function will go to 0 around the edges. And we get the usual,



What do we do for positive τ? Consider the function,



So now if we consider the integrand,



This is actually well-behaved for all z and positive τ, and 1 - nF(z) has the same poles and residues as nF(z), so we would have.



And so in total,



So we get it finally. The symmetry (even/odd) of G(τ) about the origin depends, I think, on expecting against a thermal average (or at least taking a trace of some sort). So we should have that here. The other guys would be, by analytic continuation (including the requisite implicit e∓ηt factors that come in with θ(±t)):



and the spectral function would be [using 1/(z+i0+) = P(1/z) – πiδ(z), where P stands for ‘principal value’]:



Can look at the QM/Many Particles/Identical Particles folder for more on calculating these GF’s. Just replacing state average with thermal average. The causal GF is for instance,



and then taking the Fourier transform, with the implicit convergence factor attached,



in agreement with our claim above.

**More on the Spectral Function**

We want to examine the spectral function of the single-particle operators:



So we have, from the Lehman representation of the spectral function,



Let’s consider this function in the limit that the temperature goes to zero. Then only the ground state energy survives in the sum over m (we can take the ground state energy to be zero), and we get.



(remember ξn = εn – μ). Keep in mind that |n> still indexes the many body states. But <r|n> will equal zero if |n> is anything other than a single particle state. So in effect we can just take the sum over all single particle states. Thus the interpretation is clear: A(r,r,x) is the single particle density of states (at least when T goes to 0). But we can go further and show that this is true at finite T too. Let’s start with the independent particle Hamiltonian:



Then let,



and from the Lehman representation, we have:



In the third line we recognize that |n> is restricted to differ from |m> only in that the occupation number for |n> in single particle state |j> is one greater than that for state |m>. Thus we can infer that En – Em = ξj = εj – μ. We also recognize that this collapses the sum over two independent states to just a sum over all states with the given restriction on |m>’s relationship to |n>. Now we can calculate the spectral function for any other set of creation/annihilation operators as well. First we recognize that,



(note this holds for any single particle K, k doesn’t have to index a momentum state, but can index any single particle eigenstate). Now let,



and then,



In the special case where di = ψ(r), we’d have



And indeed, we see that the density of states (per *unit* volume) is recovered when we set r = r´



So there we go! Note these ξk are the *excitations* of the *system* (which happen to be the single particle energies). So really, we might say A(r,r,x) is the density of single particle excitation states.