**Boltzman Equation w/ Single Particle Potential**

Now I’m going to go back to the original Boltzman equation.



(Ni is number of impurities) Last time we approximated the RHS with the RTA, but this treats the single particle potential scattering and two-particle potential scattering identically, and heuristically. We’d like to do a better job on the single particle potential in this file, but leave off the two-particle interaction for later.

**Classical Boltzman Equation for single particle random (impurity) potential**

So we’ll just say:



Ni is number of impurities. And the approximation which follows is still sort of heuristic, but better than RTA, and I suspect a more rigorous justification can be made, in a manner analogous to that which we will make on upcoming Fint approximation in a later file. But anyway, so the impurity potential serves to basically scatter particles from one rk state to another. And so we might just say that the rate of change of f, i.e., df/dt = (∂/∂t + k/m·∂/∂r + F·∂/∂k)f is just equal to rate at which particles are scattering into the state rk minus the rate at which they’re scattering out of the state rk. I think the coming analysis is easier to work out if I make a quick little change first. Instead of dealing with probability densities, I’m going to discretize phase space into pieces Δ3rΔ3k = (2πℏ)3, as was discussed in the very first file in the folder I think. Then we’ll let f(r,k,t) represent the probability (or equivalently, fraction) of particles being in one of these discrete chunks of phase space centered about r, k, at time t. And so for now, f is no longer a probability density, but rather just a probability. Accordingly, I’ll change the derivatives on Boltzman equation to discrete differences. Moving on, we can say,



Particles scatter via collisions with the impurities. A particle at rk can scatter out of that state by making a collision with an impurity at r. Likewise, a particle at rk´ can scatter into rk by making a collision with an impurity at r. So should be able to say:



and sum is over all k´ blocks of volume Δ3k´ for a given r. Now we say current of impurities because from the reference frame of the particle, the impurities are coming at *it*, and moreover, are coming at it with velocity v = -k/m. Let’s be a little more precise on the middle term. So,



(area would be the scattering cross-section area, σ) and similarly for the other middle term. So we can write:



where P(**rk** → **rk**´) is the probability a particle in state **rk** scatters into the state **rk**´ (really the phase space volume Δ3k´ surrounding k´). The last two terms are grouped together into a term called W, the scattering rate (can verify that it does have units of 1/time).



and likewise:



So we can write:



So now now we have for the RHS of our equation:



Filling this into our Boltzman equation, we have:



Now let’s put the equation in terms of probability densities, and scattering rate densities. So we should have f(r,k,t) → f(r,k,t)Δ3rΔ3k, which is to say we’re changing f(r,k,t) from a discrete probability distribution to a continuous probability density. And we’re going to do the same with the scattering rate W(k→k´). The rate of scattering from a given k to the Δ3k´ block surrounding k´ should be proportional to volume of k´ space available to scatter into. So we should be able to say W(k→k´) → W(k→k´)Δ3k´, where the former W(k→k´) is the discrete probability of scattering into the Δ3k´ block centered about k´, whereas the latter W(k→k´) is the scattering rate *density*, one might say. Filling these in, we have:



In the small (enough) Δ**r**, Δ**k**, and Δ3k´ limit, we can write these as derivatives and integrals. So,



This is our Boltzman equation. But we’re going to simplify a little more. W(k → k´) is the rate at which particles located at rk will scatter into rk´, per d3k´. For time-reversal symmetric processess (so no magnetic fields), this rate should be the same going forwards or backwards. This is called detailed balancing. And then our Boltzman equation would be:



where,



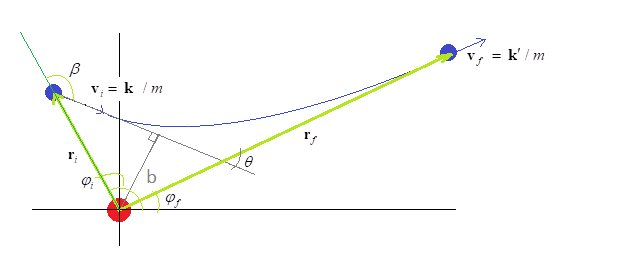
So keeping in mind our W(k→k´) was redefined as a rate density, likewise P(k→k´) is now must be a probability *density*, i.e., probability of scattering from k → k´ *per* volume Δ3k´. For most of the qualitative considerations we can stop here. But to be more quantitative, we have to put in a little more work on W. The probability (density) guy, P, is related to the differential scattering cross section converted to a probability distribution, i.e.,



where, in unfortunate notational coincidence **Ω** here is the solid angle (θ,φ) and dΩ = sin(θ)dθdφ. Pσ tells us the probability of an incoming beam with relative velocity **v**rel initially equal to **k**/m, say, being deflected into an angle **Ω** about its initial orientation (going to switch to point of view that it’s the particles scattering off of impurities rather than vice versa). The magnitude of the relative velocity is unchanged, as we know from our work in Classical Mechanics collisions stuff – basically just energy conservation. And so we have the probability of the final relative velocity **k´**/m. Should note that P(k→k´) ought to depend on k, and not just k´-k because probability of momentum transfers depends on k itself. Guess I’d say if you’re going really slow, then you’ll spend a long time in the force region of the other particle and thus probability of some large impulse, i.e., momentum transfer, would be appreciable. But if going really fast, then you won’t spend a long time in the other particle’s force region, and so experienced impulse/momentum transfer will be low. Anyway, the probability (density) distribution of the final momentum can be written in terms of Pσ via:



where **k**´(Ω) is the trajectory of the momentum after the collision. Can see this by integrating w/r to k´. So we just need to get the final velocity in terms of the initial one and the angle. A picture is helpful. General scattering process looks something like this (picture from Classical Mechanics folder – inverse square collisions). So particle starts with initial relative position vector ri and relative velocity vector vi, and exit with relative velocity vector vf. The solid scattering angle Ω is in this picture just denoted θ, and is confined to the plane defined by **r**i×**v**i.



Recall in the classical mechanics folder we examined collisions generically using momentum and energy conservation, as well as explicitly for the hard shell interaction, and the inverse square interaction. And in all cases, we found for the final velocities of the two particles:



where μ is the reduced mass, **V** is the center of mass velocity, and θiv = φi – β is the angle of the original relative velocity vector. m1,2 are masses of the particles and v1,2 the velocities. In our present case, letting 1 represent our particle and 2 represent the impurity, we’d say, having a large immobile impurity, that m2 → ∞, v2i = 0, and m1 = m, v1i = vi. In that case, μ = m, **V** = 0, **v**2f = 0 and our formula for **v**f = **v**1f simplifies to:



Allowing for an initial setup which is azimuthally rotated about the original **v**i direction by angle φ – different kind of φ from the one in the diagram above):



where is the unit vector cos(θ0v+θ)**i** + sin(θ0v+θ)**j** rotated about the cos(θ0v)**i** + sin(θ0v)**j** direction by angle φ. Note **i** and **j** are unit vectors in **r**i and **v**i plane. So now we have:



So now we have for W:



and our RHS simplifies to:



Altogether then, we have:



This can probably be derived more rigorously, starting from the very top of the file. Not going to try this here. But we will do something of the sort for the 2PI Boltzman equation.

**Relating Boltzman Equation to RTA**

One nice thing about this formula is that it allows us to get a fairly rigorous expression for 1/τ, if we still want to use the prior RTA approximation (which has the advantage of being exactly solvable). It would just be the (negative) coefficient of f(r,k,t) on the RHS, which is:



Will also note that we can go to the prior RTA approximation, then, by making the identification:



Now let’s add quantum mechanics.

**Semi-classical Boltzman Equation for single particle random (impurity) potential** We can roughly take account of fermion statistics by modifying our result to account for the fact that scattering can only take place if the final scattering state is empty. Let’s go back to the discrete version, adding the spin variable to f, and allowing for spin scattering as well.



and f is the probability of a particle having spin σ, and being in a chunk of phase space of volume Δ3rΔ3k centered around rk. And W(kσ→k´σ´) is the probability of a particle in state kσ being scattered into states within phase space volume Δ3k´ centered about k´σ´. Then we can update this formula to account for fermion statistics by adding factors that ensure the destination state is empty. The probability that none of the particles is in the state chunk centered at r,k would be 1-Nfσ(r,k), as fσ(r,k) is the probability that a *given* particle is at r,k,σ. Should probably be using N-1 rather than N, since we really want the probability that the state is occupied by particles other than the one presently scattering. But difference shouldn’t matter for systems of interest as N is very large. So,



Now convert the f and W to densities. So fσ(r,k) → fσ(r,k)Δ3rΔ3k and W(kσ→k´σ´) → W(kσ→k´σ´)Δ3k´.



Stuff cancels out,



Now since rate of scattering should be same going either way, at least for systems with Time Reversal Symmetry, we get the simplification,



And then we can take the limit that Δ → ∂,



And then we can convert the sum to an integral by taking the Δ3k → d3k limit. So then,



If it were the case that the distribution function should be independent of spin, and scattering rate should conserve spin, then this would simplify to:



And then we could sum both sides over spin variables, using f(r,k,t) = Σσfσ(r,k,t), to write:



Let’s assume this is the case, and work on the boxed equation a little more. What is W(k→k´)? Well in our continuum approximation W(k→k´)Δ3k´ is the scattering rate between states going from |k> → |k´>, where presumably these states are box normalized so that |k> = eikr/√V. Then recalling the QM folder/Time-Dependent/TDPT Scattering file, we have:



where Ni is the number of impurities. And now recalling QM folder/Scattering/Relating Scattering to TDPT file, we can say:



where the V in that squared fraction is volume, and the T-matrix is: T = V + VGV + VGVGV + … where *that* V is the single particle impurity potential. So then, we have:



Now recall we argued that Δ3k´ is just the volume in k-space taken up by a plane wave state. As discussed in the BBGKY file, this is just Δ3k´ = 2π/Lx·2π/Ly·2π/Lz = (2π)3/V. So then we have:



Filling this into the RHS of our Boltzman equation, we have:



So we end up with the same result as before,



Just with the quantum mechanical scattering cross section, rather than the classical scattering cross section.

**Putting in terms of kσ occupation numbers**

So often the Boltzman equation is not written for fσ(r,k) as a continuous probability density, but for nkσ(r) as a discrete occupation number [per Δ3rΔ3p = (2πℏ)3] function of the quantum states rkσ.



It has the properties,



using Δ3k = (2π)3/V. And averages would be done via:



So let’s make that conversion for later use. We’ll go back to the quantum continuum result.



And convert back. So let’s change integrals over k´ to sums,



But then recognize that W(kσ→k´σ´)Δ3k´ is the discrete scattering rate, which we also called W(kσ→k´σ´), so we’ll say,



And now, trivially, change variables to occupation number by multiplying everything by (2π)3N and defining (2π)3Nfσ(r,k,t). Then we have still,



In the discrete version, W is now itself the scattering rate going from the discrete state kσ → discrete state k´σ´. From quantum mechanics, the scattering rate is given by:



where Ni is the number of impurities and |k> are volume normalized states eikr/√V. And using,



we can say,



Here’s some more on the relationship between f and n,



where Δ3rΔ3p = (2πℏ)3. Last line is kind of fishy.