**Quantum Time Evolution**

Now we’ll look at the Quantum Boltzman equation. Here, instead of presuming the perturbation is small, calculating the first non-zero order correction to the distribution function, and then, evaluating the thermal expectation of our observable based on that correction to the distribution function, we leave the thermal distribution function unspecified when taking our thermal average. But we do it in a self-consistent way that thereby allows us to go beyond first order in the perturbation.

**The single particle correlation function**

In the classical approximation, we could describe the particles as having a well-defined position and momentum. And from there we examined the time development of these characteristics. We attempted this before, and modified it by introducing a scattering time based on QM. This time we’ll attempt a semi-classical approximation from the quantum end. To put ourselves in the quasi-classical regime, we consider that T ~ EF, and that length scale, time, over which single particle properties change satisfy the inequalities,



We now wish to determine an equation for the single particle correlation function (see that Many Body Correlations file in the QM folder). It looks like we’ll be presuming whatever interactions we have, they conserve spin, so that we may treat spin up/down separately, and identically. So then we may dispense with any spin subscript at all. Apropos the classical Boltzman equation, we looked for a self-consistent equation for the single particle state density function f(**r**,**p**,t). And we might attempt the same here, to look for an equation for (as worked out in the Distribution Function file):



where:



and |ψ(t)> is the many-body wavefunction as a function of time. The utility of such a function, as we also worked out in that file, is that statistical averages of any sum of single particle observables is given by:



Another asset is that while f­(r,p,t) is not itself a true probability density function, integrating w/r to r or p does give us the actual probability density distribution for p and r respectively. The easiest place to start to acquire this guy is from:



and |ψ0> is whatever state we’re starting in, which doesn’t have to be, and presumably isn’t, a thermal equilibrium state. This is pretty close to the lesser GF, which is the easiest starting point to get an equation of motion. So we’ll generalize to:



But instead of using this to get an equation for the time-development of f(r,p,t), we’re going to use it to work out an equation for the time development of a related quantity, nk(R,T), specified below. First step is to introduce ‘fast’ and ‘slow’ variables, basically relative and center-of-mass coordinates for both space and time:

 

And we’ll inttegrate the fast (relative) ones. Then what’s left will be our sought for correlation function. So we can say,



and we’ll look at the Fourier transform over the relative coordinate, (τ,**ξ**).



I guess you could say this gives us the number of particles in differential volume about **R** at time T, with energy ω and momentum **k**. But this is only a heuristic because these are not all simultaneously specifiable. More important to us would be the integral of this guy over ω.



This gives, in a fashion, the number of particles in a differential volume about **R**, with momentum **k**, at time T. Again this can’t be quite right. But it is true that the momentum distribution function is given by integration w/r to **R**, and the particle density is given by sum over **k**.



as we can see below.



So that checks out, and summing over k gives us:



So summarizing, if we integrate nk(R,T) w/r to R we get the momentum distribution function and if we ‘integrate’ it with respect to p we get the position distribution function. So we’d be rather tempted to consider np(R,T) as the simultaneous position-momentum distribution function. Of course, it cannot actually be regarded as a simultaneous position-momentum distribution function since this would violate the principles of QM. And in any event, this function isn’t even positive definite. However, it can be thought of such in a quasi-classical sense. For example, consider the one-particle quantity,



Then we can show that the average of this quantity would be:



From our work in 2nd quantization, the average of our quantity would be:



Now we have,



and the inverse Fourier transform would say,



So,



Assume that n is a slowly varying function of r, while eik´·(r-r´) is fast (because k is large, consonant with our semi-classical expectations above), and so we can just act the gradient on the exponential factor alone, because its action on n will be approximately zero. So then,



Can change spatial variables in the integrals to d3ξd3R. And can change dt´ to dτ by similar change of variables, holding T constant.



Presuming can change variables **k**´ → -**k**´, and that n will remain same. Then,



and so there we go. There’s probably an easier way.

**Equation of Motion**

It’s easiest to get the equation of motion for nk(R,T) from the lesser GF. We’ll of course recognize:



So we’ll want to develop a recursion relation for the lesser GF: G-+. We’ll start with the self-energy equations. Recall from our discussion of identical particle GF’s in the QM folder, that:



and also,



We can put the RHS of this one in terms of retarded/advanced/lesser GF’s and Σ’s, using the relationships (from same file). For instance,



and,



and also,



Then between the GF’s we have the relationships, from the same file:



So putting everything in terms of <, R, A, we will get:



Now we’ll apply the free particle differential operator i∂t – K0 (recall K0 = H0 – μ) to both sides to get:



In preparation for switching to center of mass coordiantes, we’re going to want to write equations operating on the primed variables too. If we go back to the Formal Properties file, we can repeat the ODE analysis we made on the GF’s and come to the conclusion that the same equations hold, but with i → -i. So then applying that operator to both sides of our self-energy equations, we’ll get:



Now subtract these two equations for G<, and we’ll have:



Looks like we need to pull out the single particle potential from the self-energy. So if you go back to the self-energy file, we’ll recall: ΣR = Σ-- + Σ-+, and ΣA = Σ-- + Σ+-. And if we look at the self-energies for Σ--, Σ++, Σ-+, Σ+-, we’ll observe that if we were to include a single particle potential (possibly time-dependent) into these guys, it would only show up in Σ—(i.e. -iV1) and Σ++ (i.e. +iV1). And it would take the form,



similar to the form of the tad-pole self-energy interaction term. We only have Σ—‘s in our G< equation, though. So let’s extract this from the ΣRA self-energies, and in the process recognize that the other guys have no V1 contribution to their self-energy.



and this gives us:



Now we want to get this in terms of our distribution function nkω(**R**,T). So we’ll take the Fourier transform of both sides. And we’ll note that:



and that,



and, taking some liberties with notation to change variables,



and so,



and so can say:



where we integrate by parts with the gradient term. And now it’s presumed k and ω are large and so only small values of τ and ξ will matter to the integral, and so we may expand V1 for such values,



We can integrate both sides over ω. That will go straight through the differential operators except the last. That term, by IBP, will go to zero, presuming nkω vanishes for really large |ω|. And then we’ll have:



and so it follows that:



and therefore, our most general form for the Boltzman equation will be:



where:



And I don’t want to work on this anymore. So far this equation is exact and nothing regarding the irreversible processes has been added here so far. In that sense it is like the classical Bolztman equation:



which up to this point, also was exact. But like with the classical equation, we made approximations on the RHS which implicitly modelled irreversible processes [one step along the way was modelling the many body force term as a single body term which also made the equation self-consistent]. Accordingly we can do the same with our quantum equation. The RHS can be simplifed, with some additional approximations regarding the structure of the GF’s, and Σ’s, to make it a self-consistent equation single body equation – which would implicitly model irreversible processes. And then some more approximations can be made to reduce the RHS to the typical scattering form found in the classical approximation, though with the generalization that we need not assume equilibrium statistics.