**Boltzman Equation w/ Single Particle Potential**

Now let’s revisit our equation,



and try to solve it.

**Equilibrium Solution to Boltzman Random 1PI equation**

Let’s pause and consider equilibrium solutions to this equation. This would be any time-independent solution. As with last time, we’ll presume only conservative forces present here, so: **F** = -∇φ. Then generic solutions to the LHS would, like with RTA, be of the form:



For instance, with the former ansatz,



Apropos the RHS, any local function of energy, of the form, say,



will satisfy the RHS since the collisions are elastic, and so k and k´ have same kinetic energy. So global equilibrium (true equilibrium) solutions would be, according to this equation, something like:



There is nothing in this equation alone to force fto take a form such as: f ~ e-βε, etc. We’ll see that we need to include the two-particle interaction to get that restriction. We also explicitly see that the distribution function won’t change if it’s just responding to its own forces. But it will change if add a non-conservative force field to it, and if we allow T, or μ to depend on position. So we’d basically like to see how it changes if we do so.

**Non-Equilibrium Solution to Boltzman Random 1PI equation**

Presuming we knew W, how would we solve it? We probably can’t solve it exactly, but we can obtain the first order response solutions, without too much difficulty, and this is what we really want. So let’s say we have a setup with a spatially constant (though maybe not temporally) temperature gradient, chemical potential gradient, and non-conservative force Fnc. We’d like steady state solution to first order in these gradients. We may suppose that the solution deviates from the local equilibrium by just a small amount. The small amount is the part that describes the entropic currents that must flow in response to their not being global equilibrium. So something like,



where ε = k2/2m + φ(r,t) and β(r,t) = 1/T(r,t). Then let’s say f = fleq + δf. Plugging this in, and remembering that since the collisions are presumed elastic, fleq annihilates the RHS, we get:



Now gotta take all those derivatives of fleq. Going to write it in a way that easily generalizes to the semi-classical case:



So now,



Now to first order we should have that δf will be proportional to the temperature, chemical potential, force potential gradients, and so we’d anticipate that acting (k/m)∂/∂r + F(r)·∂/∂k on it will just introduce more gradients and so make it second order. So we can neglect that term, to first order, which leaves us with:



Defining the operator, :



We can write:



Alas, we still have an integral equation, but at least it’s not an integro-differential equation – sort of. One might hope that the solution looks similar to what we found using the RTA. Let’s specialize to the time-independent case, and φ = 0 case. Then our equaiton is:



Then the RTA solution was:



And so I’ll guess a solution of the same form,



where ε = k2/2m and τtr(ε) is a generalization of τ(k), which we’ll use to fit the equation. So then the integrand of δf (sans the W term) is:



Plugging this into our equation, we’ll note that everything except for the **k** can come outside the integral because **k** and **k**´ both have the same energy.



Now I think we can say that:



will point along the direction of **k** because the scattering function W(k→k´) should be symmetric about **k**. Since that’s the case, we can explicitly isolate that component, and write, remembering that **k** and **k**´ have the same magnitude,



So then we have:



And solving for τtr(ε), we have:



Looks like we now have an explicit formula for the transport scattering lifetime 1/τtr. Yay 😊. And our (first order) solution is:



where ε = k2/2m of course. Might note that since fleq,σ is purely a function of ε, we could also write this, to first order, as:



which is to say the perturbation has simply made a change to the energy levels. It looks like the change in energy is something like the energy acquired by the electron over the course of the mean free time. But writing it this way does obscure the fact that the force field, and other gradients, are driving the charges *out* of equilibrium – they are not settling into an equilibrium distribution with the field and gradients. Still, it can provide a useful picture.

**Example: hard sphere**

Let’s specialize to hard sphere interaction, dσ/dΩ = σ/4π. And let’s align the z about which Ω rotates with **k**. Then we have:



Well I was expecting something else. But that’s a nice simple result. Let’s do the Coulomb scattering cross-section instead. In the Classical Mechanics folder we found the Coulomb scattering cross section to be:



where kE = e2/4πε0, T = k2/2m, and σ = πbmax2. bmax would basically be half the distance between neighboring atoms. So,



Okay gonna stop there. Well…say θmin is small, then I think 1/x2 dominates ln(x). And further 1/sin2 ~ cot2. And so,



Well that’s different. At least it does show that as k → ∞, τtr(ε) → ∞. This makes sense because when k is super large, all scattering will roughly be small angle type, and such small deflections would take a longer time to effectively randomize the velocity out of its proclivity to follow the external field. This apparently compensates for the fact that larger k also means greater rate of collisions. Let’s compare this to the bare (thermal) relaxation time,



This, in contrast, gets smaller with large k, which also makes sense because larger k means greater rate of collisions. And so the mean free time should get smaller.

So transport relaxation time is generally longer since some of the collisions will scatter the particle into directions close to the original direction, which wouldn’t change the distribution hardly at all, and so it takes more collisions to truly relax back to equilibrium than it would otherwise.

**Semi-classical Solution to 1PI Boltzman equation**

We can extend our analysis to the semi-classical case.



We can also propose a solution of the form,



We can plug this into our equation,



Now gotta take all those derivatives of fleq,σ. Can see that we’ll get the same thing as before,



So now,



Now we should have that δfσ will be proportional to the temperature, chemical potential, force potential gradients, and so we’d anticipate that acting (k/m)∂/∂r + F(r)·∂/∂k on it will just introduce more gradients and so make it second order. So we can neglect that term, which leaves us with:



Defining the operator, :



We can write:



I’ll specialize again to the time-independent, and φ = 0 case,



and propose a solution of similar form as proposed above,



where ε = k2/2m and τtr,σ(ε) is a generalization of τσ(k), which we’ll use to fit the equation. So then the argument of δf, sans the W, is:



Plugging this into our equation, we’ll note that everything except for the **k** can come outside the integral because **k** and **k**´ both have the same energy.



I’m going to presume that the momentum and spin parts of the rate are separable,



And we should still be able to say,



And so,



Filling this in,



Comparing both sides, we have:



which is a matrix equation to solve for the spin-dependent transport relaxation time. I guess it’s not worth pursuing further in general. But suppose that there were no spin part. Then we’d get the usual result for τtr(ε),



with dσ/dΩ being the quantum mechanical scattering cross-section, per our analysis in the previous file. And our (first order) solution is just as the classical one, but with the analogous fleq.



where ε = k2/2m of course.